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The modes of an ultra-cold strongly magnetized charged Bose gas

V. Kowalenko[a]

School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia

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Abstract. The modes of a strongly magnetized charged Bose gas are presented for ultra-low temperatures. For longitudinal oscillations propagating parallel to the magnetic field the dispersion relation is found to be dominated by the one-dimensional field-free plasmon dispersion relation as found by Alexandrov, Beere and Kabanov recently in reference [1], while for propagation perpendicular to the magnetic field they are found to be influenced by the cyclotron motion of the particles. Dispersion relations for these modes known as Bernstein modes are given near the cyclotron frequency and its first two harmonics. The dispersion relations for transverse modes in the system are then presented for the cases of photon propagation perpendicular and parallel to the direction of the magnetic field.

PACS. 05.30.Jp Boson systems – 74.20.Mn Nonconventional mechanisms (spin fluctuations, polarons and bipolarons, resonating valence bond model, anyon mechanism, marginal Fermi liquid, Luttinger liquid, etc.) – 02.30.Gp Special functions

1 Introduction

Aside from the fact that the magnetized charged Bose gas (CBG) is still an unsolved fundamental problem in many body physics, the system has attracted much interest lately because of its role in the bipolaron theory of high temperature superconductivity [2,3]. Recently, the dielectric properties of the CBG were presented in a weak external magnetic field at $T = 0$ K where quantum effects dominate in the system [4]. In actual fact, these results were derived under the condition that $|\omega_c/\omega_p| < 1$, where ω_c $(= eB/mc)$ and ω_p $(= \sqrt{4\pi ne^2/m})$ are, respectively, the cyclotron and plasma frequencies. Thus, not only are these results valid for a weakly magnetized system, they are also valid for the strong magnetic field case provided the plasma frequency is much greater than the cyclotron frequency. For this latter case, one need not be at $T = 0$ K, since the Bose distribution function at sufficiently low temperatures can be approximated by its $T = 0$ K form. Thus, the results in reference [4] can be regarded as the high density limit low temperature regime for the strongly magnetized CBG.

One of the most striking features of high temperature superconductors is their high upper critical fields, which appear to diverge as the temperature is lowered below T_c. For example, measurements with overdoped Tl-Ba-Cu-O [5] and Bi-Sr-Cu-O [6] have exposed the diverging nature of $H_{c2}(T)$ from their T_c of approximately 20 K down to millikelvin temperatures reaching values of 3×10^5 G (30 T) and 2×10^5 G, respectively. For the case of the superconducting materials where $T_c > 60$ K, the situation is more difficult because not only are the H_{c2} values higher, but the in-plane superconducting transition is known to exhibit pronounced broadening in a magnetic field, with the top of the transition having a much weaker field dependence than in the region near the bottom [7–9]. Whilst it, therefore, has been difficult to determine $H_{c2}(T)$ for these superconductors experimentally, Alexandrov *et al.* [10] have proposed a method for extrapolating the values of the resistive upper critical field up to $H_{c2} \approx 2.3 \times 10^6$ G and $T/T_c \approx 0.35$ based on their own theory of Bose-Einstein condensation in a magnetic field [11]. If electrons in these materials couple very strongly to the lattice, then Bose particles in the form of bipolarons can form. Thus it becomes necessary to study the CBG in a strong magnetic field at low temperatures, but, unfortunately, for the fields given above the cyclotron frequency is considerably greater than the plasma frequency, thereby rendering the results in reference [4] inapplicable.

The present paper aims to use a novel asymptotic expansion, first presented in reference [16], for the Kummer function that appears in the various dielectric response functions for the system, thereby enabling the evaluation of the dispersion relation for longitudinal and transverse modes for a CBG with $|\omega_p/\omega_c| < 1$. These properties have remained elusive in the past because of the highly anisotropic nature of the system. To evaluate these dispersion relations we shall use the $T = 0$ K form of the Bose distribution function, which is valid for high T_c superconductors provided the magnetic fields are high. In

[a] e-mail: vak@physics.unimelb.edu.au

line with the opening paragraph of this paper, the regime of $|\omega_p/\omega_c| < 1$ can also be regarded as the weakly magnetized ultra-low density limit, but for this situation, the results presented here are restricted to extremely low temperatures close to $T = 0$ K.

This paper is arranged as follows. In Section 2 the dispersion relations for longitudinal or collective modes are presented. Here not only is the dispersion relation for plasmons presented, but also the dispersion relations for Bernstein modes [12], which propagate perpendicular to the direction of the magnetic field. Dispersion relations for these modes near the cyclotron frequency and near its first two harmonics are given in addition to the dispersion relationship for the general anisotropic case. In Section 3 the dispersion relations corresponding to photon propagation perpendicular and parallel to the magnetic field are presented. The results of this work are summarized in the concluding section.

2 Collective modes

Realizing that the properties of a strongly magnetized CBG have never been adequately evaluated, Alexandrov *et al.* have recently sought to develop the theory of the CBG in an ultrahigh magnetic field at ultracold temperatures [1]. In this work they eventually obtained the plasmon dispersion relation of

$$\omega^2 = \frac{\hbar^2 q_z^4}{4m^2} + \frac{q_z^2}{q^2}\,\omega_p^2\,e^{-\gamma(q_\perp)}, \qquad (1)$$

where q_z and $q_\perp$ are respectively the components of the wavenumber parallel and perpendicular to the magnetic field arising from the anisotropy of the system and $\gamma(q) = \hbar q^2/2m\omega_c$. For $q_\perp = 0$ this reduces to the field-free Foldy spectrum [13], which is consistent with the classical behavior that a strongly magnetized plasma behaves as if it were a one-dimensional gas [14]. In the present work a slightly more general form which reduces to equation (1) for $q_\perp = 0$ and $|\omega_c| \to \infty$ will be derived *via* the random phase approximation (RPA) method as opposed to the Bogoliubov de Gennes approach used by Alexandrov *et al.* While it is well-known that another longitudinal mode with a frequency near the cyclotron frequency or a harmonic of it can propagate principally in directions perpendicular to the magnetic field, *i.e.*, $q_z \approx 0$, this mode known as a Bernstein mode [12,14] cannot be derived from equation (1), but can be derived *via* the RPA approach.

The longitudinal dielectric response function for arbitrary distribution function has been derived for the magnetized CBG *via* a self-consistent, second-quantized RPA approach in references [15,16]. In the first of these references, the conductivity tensor is evaluated while in the second reference the polarization tensor is evaluated. The advantage of the polarization tensor approach is that it is independent of gauge and therefore, must satisfy symmetry properties such as charge conjugation and the Onsager relation. The conductivity tensor derived from the more elegant polarization tensor was found not to be identical to the result obtained by Hore and Frankel [15], which lacked additional terms pertaining to the transverse modes/properties of the magnetized CBG. Thus, although Hore and Frankel were able to provide the correct form for the longitudinal dielectric response of the system, they would not have been able to carry out a correct study of its transverse properties. In addition, while the longitudinal dielectric response function was correct, they were primarily interested in studying the weak field or high density limit rather than the strong field or low density limit. Specifically, their study was applicable to $|\omega_c/\omega_p| \ll 1$, whereas this paper is concerned with $|\omega_c/\omega_p| > 1$, which represents a different asymptotic regime. The properties of the magnetized CBG for $|\omega_c/\omega_p| < 1$ are addressed in reference [4], where it is shown that many of the results obtained by Hore and Frankel are, indeed, incorrect.

Introducing the $T = 0$ K distribution function into this result yields

$$\epsilon_L(q_z, q_\perp, \omega) = 1 + \frac{m\omega_p^2}{\hbar q^2}\frac{e^{-\gamma(q_\perp)}}{\omega_c} \times\Big[S\Big(\frac{\hbar q_z^2/2m + \omega}{\omega_c}, \gamma(q_\perp)\Big) + S\Big(\frac{\hbar q_z^2/2m - \omega}{\omega_c}, \gamma(q_\perp)\Big)\Big], \qquad (2)$$

where $q^2 = q_z^2 + q_\perp^2$. In equation (2) $S(\alpha, x)$ represents

$$S(\alpha, x) = \sum_{k=0}^{\infty} \frac{z^k}{(k+\alpha)\,k!}\cdot \qquad (3)$$

It should be noted that in equation (2), ω_c has been assumed to be positive, but if it is negative, then it should be replaced by $|\omega_c|$, which is a consequence of the charge conjugation property of the polarization tensor [16]. In addition, since all modes in the magnetized CBG at $T = 0$ K are zero-damped, the Landau prescription of replacing ω by $\omega + i\eta$ [17] is not required here.

As mentioned above, in reference [10] Alexandrov *et al.* have extrapolated the $H_{c2}(T)$ curve to $0.35T_c$ to obtain a critical field of 230×10^4 G, which in turn yields a cyclotron frequency of 8.5×10^{13} rad s^{-1} for a $2e$ particle. For these conditions $\gamma(q_\perp)$ is very small unless $q_\perp$ is of the order of 10^6 cm^{-1}. However, although ω_c is large, ω/ω_c need not necessarily be small because it may be a Bernstein mode. Therefore, to analyze the response functions for large ω_c, we require a small $|x|$ expansion for $S(\alpha, x)$, that is general for all values of α. Such a small $|x|$ expansion has been obtained in reference [18]. This expansion, which is more accurate than just using the first few terms of the series expansion for $S(\alpha, x)$ given by equation (3), is

$$S(\alpha, x) \sim \frac{1}{\alpha} + x\,e^x \sum_{m=0}^{\infty}(-\alpha)^m \sum_{k=0}^{\infty} \frac{\Gamma(k+m+2)}{\Gamma(m+2)}\frac{c_k(x)}{(x+1)^{k+m+2}}\cdot \qquad (4)$$

The $c_k(x)$ in equation (4) are the same polynomials that appear in the large $|\alpha|$ expansion for $S(\alpha, x)$ used to evaluate the dielectric properties in the weakly magnetized CBG [4].

Noting that $S(\alpha, x)$ is related to the incomplete gamma function, Kowalenko and Taucher [19] have carried out a detailed investigation of the large $|\alpha|$ expansion for the latter function in which several general expressions for the coefficients of the $c_k(x)$ have been calculated. For example, by denoting c_k^j as the coefficient of x^j in each $c_k(x)$, they found that

$$c_k^1 = (-1)^k / k!, \qquad (5)$$

and

$$c_k^2 = (-1)^k \Big(\frac{2^{k-1}-1}{k!} - \frac{1}{(k-1)!} \Big). \qquad (6)$$

Introducing these general expressions into equation (4) yields the following small $|x|$ expansion for $S(\alpha, x)$:

$$S(\alpha,x) \sim x e^x \left[\sum_{k=1}^{3} \frac{x^{k-1}}{(k-1)!\, f(x,k)} - \sum_{k=0}^{1} x \frac{\partial^k}{\partial x^k} \frac{1}{f(x,1)} - \frac{x^2}{2} \sum_{k=1}^{4} \frac{2^{k-1}}{(k-1)!} \frac{\partial^{k-1}}{\partial x^{k-1}} \frac{1}{f(x,1)} - x^2 \frac{\partial}{\partial x} \frac{1}{f(x,2)} + x^2 \sum_{k=1}^{3} \frac{1}{(k-1)!} \frac{\partial^k}{\partial x^k} \frac{1}{f(x,1)} + O(x^3) \right] + \frac{1}{\alpha}, \qquad (7)$$

where $f(x,k) = (x+k)(x+\alpha+k)$. If $|x| \ll |\alpha+1|$, then equation (7) can be expanded further to give the first few terms in the power series expansion for $S(\alpha, x)$.

If equation (7) is introduced into equation (3), then the longitudinal dielectric response function is given by the asymptotic form of

$$\epsilon_{\rm L}(q_z, q_\perp, \omega) \sim 1 + \frac{q_z^2}{q^2} \frac{\omega_{\rm p}^2 \exp(-\gamma(q_\perp))}{(\hbar^2 q_z^4/4m^2 - \omega^2)} + \frac{q_\perp^2}{q^2} \frac{\omega_{\rm p}^2}{(\omega_{\rm c}^2(1+\gamma(q))^2 - \omega^2)} \times \left(1 - \gamma(q_\perp) + \gamma(q_z) \frac{(1-\gamma(q_\perp))}{(1+\gamma(q_\perp))} + \frac{\gamma(q_\perp)\gamma(q_z)}{(1+\gamma(q_\perp))^2} \right) + \frac{q_\perp^2}{q^2} \frac{2\omega_{\rm p}^2 \omega_{\rm c}^2 (1+\gamma(q))^2}{(\omega_{\rm c}^2(1+\gamma(q))^2 - \omega^2)^2} \frac{\gamma(q_\perp)}{(1+\gamma(q_\perp))} + \frac{q_\perp^2}{q^2} \times \left(1 + \frac{\gamma(q_z)}{2+\gamma(q_\perp)} \right) \frac{\omega_{\rm p}^2 \gamma(q_\perp)}{(\omega_{\rm c}^2(2+\gamma(q))^2 - \omega^2)} + O\left(q_\perp^6\right). \qquad (8)$$

Equation (8) has been determined by considering only the terms up to $O(x^2)$ in the asymptotic expansion given by equation (7). That is, terms of $O(x^2)$ derived from equation (6) in the square-bracketed expression of equation (7) have not been displayed. These will be used shortly for the analysis of collective modes perpendicular to the magnetic field. The dispersion relation for longitudinal modes is obtained from $\epsilon_{\rm L}(q,\omega) = 0$. In setting equation (8) equal to zero, it can be seen immediately that the plasmon dispersion relation obtained by Alexandrov *et al.* in reference [1], *viz.*, equation (1) here, can only be obtained if $q_\perp = 0$, although for this case the factor $\exp(-\gamma(q_\perp))$ in equation (1) should be set equal to unity. Strictly speaking, the exponential factor should be expanded in powers of $\gamma(q_\perp)$ since all the other terms in equation (8) have been obtained under this condition. Thus, the dispersion relation given by equation (1) corresponds to $q_\perp = 0$.

For the propagation of collective modes perpendicular to or across the magnetic field ($q_z = 0$), the longitudinal dielectric response function reduces to

$$\epsilon_{\rm L}(q,\omega) \sim 1 + \frac{(1-\gamma(q)-\gamma(q)^2/2)\,\omega_{\rm p}^2}{\omega_{\rm c}^2(1+\gamma(q))^2 - \omega^2} + \frac{2\gamma(q)(1+\gamma(q))\,\omega_{\rm p}^2\omega_{\rm c}^2}{(\omega_{\rm c}^2(1+\gamma(q))^2-\omega^2)^2} + \frac{\gamma(q)\,\omega_{\rm p}^2}{\omega_{\rm c}^2(2+\gamma(q))^2-\omega^2} + \frac{\gamma(q)\,\omega_{\rm p}^2\omega_{\rm c}^2/2}{(\omega_{\rm c}^2(3+\gamma(q))^2-\omega^2)^2} + \frac{2\gamma(q)^2(2+\gamma(q))\,\omega_{\rm p}^2\omega_{\rm c}^2}{(\omega_{\rm c}^2(2+\gamma(q))^2-\omega^2)^2} + \frac{8\gamma(q)^2(\gamma(q)+1)^3\,\omega_{\rm p}^2\omega_{\rm c}^6}{(\omega_{\rm c}^2(1+\gamma(q))^2-\omega^2)^4} - \frac{4\gamma(q)^2(1+\gamma(q))\,\omega_{\rm p}^2\omega_{\rm c}^4}{(\omega_{\rm c}^2(1+\gamma(q))^2-\omega^2)^3}, \qquad (9)$$

where the terms due to equation (6) in the asymptotic expansion for $S(\alpha, x)$ have now been included. If $\hbar q^2/2m\omega_{\rm c}$ or $\gamma(q)$ is assumed to be small, *i.e.*, much less than unity, then the dispersion relation obtained after setting equation (9) equal to zero is

$$\omega^2 \approx \omega_{\rm c}^2\,(1+\gamma(q))^2 + \omega_{\rm p}^2 - 2\gamma(q)\,\omega_{\rm c}^2 \times \left(\frac{1+\omega_{\rm p}^2/2\omega_{\rm c}^2 + \omega_{\rm p}^4/6\omega_{\rm c}^4}{1-4\gamma(q)\omega_{\rm c}^2/\omega_{\rm p}^2} \right) + \cdots. \qquad (10)$$

Equation (10) has been evaluated by carrying out a perturbational analysis of solution to the dispersion relation obtained by putting $\gamma(q)$ equal to zero. For $\omega_{\rm p} \ll \omega_{\rm c}$, this zero-damped longitudinal mode is close to the cyclotron frequency and hence, represents a Bernstein mode [12]. On the other hand, putting $\omega_{\rm c} = 0$ in equation (10) yields the field-free Foldy spectrum [13] since the first couple of terms in equation (7) appear in the expansion for the weakly magnetized case [4].

For ω close to $(2+\gamma(q))\,\omega_{\rm c}$, one can obtain another solution to the dispersion relation given by

$$\omega^2 \approx \Big(2+\gamma(q)\Big)^2 \omega_{\rm c}^2 + \omega_{\rm p}^2\,\gamma(q) \Big(1 + \frac{\omega_{\rm p}^2}{\alpha_0} \Big(1-\gamma(q)\Big) - \frac{\hbar q^2 \omega_{\rm p}^2\,\omega_{\rm c}}{m\,\alpha_0^2} \Big) + \ldots, \qquad (11)$$

where $\alpha_0 = 3\omega_{\rm c}^2 + \hbar q^2\omega_{\rm c}/m + \hbar q^2\omega_{\rm p}^2/2m\omega_{\rm c}$. This solution represents another zero-damped collective mode propagating across the magnetic field except on this occasion it is near a harmonic of the cyclotron frequency. Close to $(3+\gamma(q))\,\omega_{\rm c}$ yet another zero-damped solution to the longitudinal dispersion relationship can be found, which is

$$\omega^2 \approx \left(3\omega_{\rm c} + \frac{\hbar q^2}{2m}\right)^2 + \frac{\hbar^2 q^4 \omega_{\rm p}^2}{8m^2\omega_{\rm c}^2} + \frac{\hbar^2 q^4 \omega_{\rm p}^4}{64 m^2 \omega_{\rm c}^4} + \cdots . \quad (12)$$

Close to the $n-1$-th harmonic, we can use the preceding material to conjecture that the longitudinal dielectric response function is approximately

$$\epsilon_{\rm L}(q,\omega) \sim 1 + \frac{\omega_{\rm p}^2}{((\gamma(q)+1)^2\omega_{\rm c}^2 - \omega^2)} + \frac{\gamma(q)^{n-1}\,\omega_{\rm p}^2}{\Lambda(n-1)((\gamma(q)+n)^2\omega_{\rm c}^2-\omega^2)}\,, \quad (13)$$

where $\Lambda(n-1)$, which can only be determined by considering higher order terms in the asymptotic expansion given by equation (7), is either equal to $n-1$ or more likely $(n-1)!$. The solution to the dispersion relation close to this harmonic is

$$\omega^2 \approx \left(n\omega_{\rm c} + \frac{\hbar q^2}{2m}\right)^2 + \frac{\gamma(q)^{n-1}}{\Lambda(n-1)}\left(1 + \frac{\omega_{\rm p}^2}{(n^2-1)\omega_{\rm c}^2}\right) + \cdots . \quad (14)$$

So far, we have concentrated on the modes parallel and perpendicular to the the magnetic field. We have seen that plasmon modes propagate parallel to the magnetic field while perpendicular to the field Bernstein modes propagate, whose frequencies are dominated by the cyclotron frequency or harmonics of it. It has also been seen as the magnetic field is lowered that these modes merge into the plasmon dispersion relation eventually yielding the field-free Foldy spectrum [13].

We now consider the general anisotropic case where both q_z and $q_\perp$ are not equal to zero. To first order in $\hbar q^2/m\omega_{\rm c}$, *i.e.*, for very strong magnetic fields, equation (8) becomes

$$\begin{aligned}\epsilon_{\rm L}(q_z,q_\perp,\omega) \approx 1 &+ \frac{q_z^2}{q^2}\,\frac{\omega_{\rm p}^2\,(1-\hbar q_\perp^2/2m\omega_{\rm c})}{(\hbar^2 q_z^4/4m^2-\omega^2)} \\ &+ \frac{q_\perp^2}{q^2}\,\frac{\omega_{\rm p}^2\,(1+\hbar q_z^2/2m\omega_{\rm c}-\hbar q_\perp^2/2m\omega_{\rm c})}{(\omega_{\rm c}^2\,(1+\hbar q^2/m\omega_{\rm c})-\omega^2)} \\ &+ \frac{q_\perp^2}{q^2}\left(\frac{\hbar q_\perp^2}{m\omega_{\rm c}}\right)\frac{\omega_{\rm p}^2\,\omega_{\rm c}^2}{(\omega_{\rm c}^2(1+\hbar q^2/m\omega_{\rm c})-\omega^2)^2} \\ &+ \frac{q_\perp^2}{q^2}\left(\frac{\hbar q_\perp^2}{2m\omega_{\rm c}}\right)\frac{\omega_{\rm p}^2}{(4\omega_{\rm c}^2(1+\hbar q^2/2m\omega_{\rm c})-\omega^2)}\,. \quad (15)\end{aligned}$$

By taking the $\hbar q^2/m\omega_{\rm c} \to 0$ limit and then setting this result equal to zero, one finds that the dispersion relation for collective modes reduces to a quartic, whose solutions are

$$\omega^2 = \frac{1}{2}\left(\omega_{\rm p}^2 + \omega_{\rm c}^2 + \frac{\hbar^2 q_z^4}{4m^2} \pm\sqrt{\left(\frac{q_z^2\omega_{\rm p}^2}{q^2} + \frac{\hbar^2 q_z^4}{4m^2} - \frac{q_\perp^2\omega_{\rm p}^2}{q^2} - \omega_{\rm c}^2\right)^2 + 4\,\frac{q_z^2 q_\perp^2}{q^4}\,\omega_{\rm p}^4}\right). \quad (16)$$

For $\omega_{\rm p}^2 \ll \omega_{\rm c}^2$, the two branches corresponding to the positive and negative square roots in equation (16) become

$$\omega^2 \approx \omega_{\rm c}^2 + \frac{q_\perp^2\omega_{\rm p}^2}{q^2} + \frac{q_z^2 q_\perp^2}{q^4}\,\frac{\omega_{\rm p}^4}{\beta\,\omega_{\rm c}^2} + \cdots\,, \quad (17)$$

and

$$\omega^2 \approx \frac{q_z^2\omega_{\rm p}^2}{q^2} + \frac{\hbar^2 q_z^4}{4m^2} - \frac{q_z^2 q_\perp^2}{q^4}\,\frac{\omega_{\rm p}^4}{\beta\,\omega_{\rm c}^2} + \cdots\,, \quad (18)$$

where $\beta = 1-\gamma(q_z)^2$. In the limit $|\omega_{\rm p}/\omega_{\rm c}| \to 0$, equation (18) reduces to the result obtained by Alexandrov *et al.* [1], *viz.* equation (1). For $\omega_{\rm p}^2 \gg \omega_{\rm c}^2$, equation (16) yields

$$\omega^2 \approx \frac{1}{2}\left(\omega_{\rm p}^2 + \omega_{\rm c}^2 + \frac{\hbar^2 q_z^4}{4m^2} \pm \omega_{\rm p}^2 \pm\left(\frac{\hbar^2 q_z^4}{4m^2} - \omega_{\rm c}^2\right)\left(\frac{q_z^2 - q_\perp^2}{q^2} + \cdots\right)\right). \quad (19)$$

The above results can now be used to carry out a perturbational analysis on the dispersion relation obtained by setting the longitudinal response function given by equation (15) equal to zero. This yields a sextic polynomial, two of whose roots represent perturbations of the above results while the remaining solution is that obtained near the first harmonic of the cyclotron frequency. We begin with the solution near this first harmonic, which is only valid provided $q_\perp \neq 0$. By concentrating on the denominator in the final term of equation (15), one obtains

$$\begin{aligned}\omega^2 = 4\omega_{\rm c}^2&\left(1 + \frac{\hbar q_\perp^2}{2m\omega_{\rm c}}\right) \\ &+ \omega_{\rm p}^2\left(\frac{\hbar q_\perp^2}{2m\omega_{\rm c}}\right)\left(\frac{q_\perp^2}{q^2}\right)\left(1 + \frac{\omega_{\rm p}^2}{4\omega_{\rm c}^2} \pm \frac{q_\perp^2}{3q^2}\,\frac{\omega_{\rm p}^2}{4\omega_{\rm c}^2}\right) \\ &+ O\left(\left(\frac{\hbar q^2}{2m\omega_{\rm c}}\right)^2\right). \quad (20)\end{aligned}$$

It should be emphasised that in deriving equation (20) no assumption other than $\gamma(q) \ll 1$ has been made. That is, no assumption concerning the size of $\omega_{\rm p}$ to $\omega_{\rm c}$ has had to be invoked.

From equation (16) it can be seen that taking the positive square root in equation (15) yields the branch near the cyclotron frequency. However, rather than deal with the radical in equation (15), we shall consider perturbations close to where the selected value for ω^2 will yield negative unity as the dominant term in the denominator of the third term on the rhs of equation (15), namely, $\omega^2 = \omega_c^2(1 + \hbar q^2/m\omega_c) + q_\perp^2\omega_p^2/q^2 + \chi$ where χ is assumed to be small and $q_\perp^2\omega_p^2 \ll q^2\omega_c^2$. We also need to ensure that the next term in the longitudinal response function is small for this branch, which in turn means that $|\hbar q^2/2m\omega_c| \ll q_\perp^2\omega_p^2/q^2\omega_c^2$. To ensure that the final term is small, we must have $|\hbar q_\perp^2/6m\omega_c| \ll q^2\omega_c^2/q_\perp^2\omega_p^2$. This appears to contradict the condition for smallness of the previous term, but in reality means that $|\hbar q^2/m\omega_c|$ must be much smaller than the minimum of ω_p^2/ω_c^2 and ω_c^2/ω_p^2. Finally, to ensure that that the second term on the rhs of equation (15) is small for these values of ω, we must have $q_z^2\omega_p^2 \ll q^2\omega_c^2$.

When these conditions are satisfied, one finds that the dispersion relation near the cyclotron frequency becomes

$$\omega^2 \approx \omega_c^2\left(1+\frac{\hbar q^2}{m\omega_c}\right) + \frac{q_\perp^2\omega_p^2}{q^2} + \frac{q_\perp^2 q_z^2\,\omega_p^4}{q^4\,\omega_c^2} + \frac{q_\perp^2\,\omega_p^2}{q^2} \times\left(\frac{\hbar q_z^2}{2m\omega_c} - \frac{\hbar q_\perp^2}{2m\omega_c} - \left(\frac{\hbar q^2}{m\omega_c}\right)\frac{\omega_c^2}{\omega_p^2}\right) + \cdots\,. \quad (21)$$

The result given by equation (18) does not contain all the first order terms from the longitudinal dielectric response function, so we now consider $\omega^2 = q_z^2\omega_p^2/q^2 + \hbar^2 q_z^4/4m^2 + \chi$, where χ is assumed to be small. To ensure that the various terms appearing in equation (3) remain small, we must take $\omega_p^2 \ll \omega_c^2$ again. Then, the dispersion relation to first order becomes

$$\omega^2 \approx \frac{q_z^2\omega_p^2}{q^2}\left(1 - \frac{\hbar q_\perp^2}{2m\omega_c} - \frac{q_\perp^2\omega_p^2}{q^2\omega_c^2}\right) + \frac{\hbar^2 q_z^4}{4m^2} + \cdots\,, \quad (22)$$

where it can be seen that the branch is zero for propagation perpendicular to the magnetic field ($q_z = 0$). Once again, in the ultra-strong field limit of $|\omega_c| \to \infty$, equation (22) reduces to the result obtained by Alexandrov *et al.* [1], which appears as equation (1) here.

3 Transverse modes

For photon propagation parallel to the magnetic field, however, the transverse dielectric response functions are conveniently described in terms of left and right circularly polarized forms, which from reference [4] are given by

$$\epsilon_l(q,\omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_p^2}{\omega^2}\left(\frac{\omega_c}{\omega - \omega_c - \hbar q^2/2m}\right), \quad (23)$$

and

$$\epsilon_r(q,\omega) = 1 - \frac{\omega_p^2}{\omega^2} + \frac{\omega_p^2}{\omega^2}\left(\frac{\omega_c}{\omega + \omega_c + \hbar q^2/2m}\right). \quad (24)$$

For $|\omega - \omega_c| \gg \hbar q^2/2m$, these dielectric response functions reduce to those for an electron plasma in the cold plasma limit [14]. The dispersion relations for left and right circularly polarized modes are obtained by setting both equations equal to q^2c^2/ω^2. The resulting equations have been solved in the weak field limit in reference [4]; so here, we shall be concerned with the large ω_c solutions for the modes.

From the above the dispersion relations for circularly polarized modes become

$$\omega^3 \pm (\omega_c + \hbar q^2/2m)\,\omega^2 - (\omega_p^2 + q^2c^2)\,\omega \mp \omega_c\, q^2c^2 \mp (\omega_p^2 + q^2c^2)\hbar q^2/2m = 0, \quad (25)$$

where '+' denotes right circularly polarized modes and '−' left circularly polarized modes. For large $|\omega_c|$, the first two terms on the lhs of both equations dominate and thus the dispersion relations can be found by perturbing around the solutions obtained when setting these terms equal to zero. For left circularly polarized modes, one eventually obtains

$$\omega = \omega_c + \frac{\hbar q^2}{2m} + \frac{\omega_p^2\,\omega_c}{(\omega_c + \hbar q^2/2m)^2} + \cdots\,, \quad (26)$$

while for right circularly polarized modes, one obtains the negative of this result. Thus for circularly polarized modes there is only one dispersion relation that arises in the vicinity of the fundamental of the cyclotron frequency, which is unlike the situation for the collective modes of the strongly magnetized CBG where dispersion relations in the vicinity of harmonics of the cyclotron frequency can also be obtained. Resonances for these modes occur whenever equations (23, 24) yield infinities. Thus for left circularly polarized modes a resonance occurs at $\omega = \omega_c + \hbar q^2/2m$ while equation (24) possesses a resonance at negative values of this frequency. Cut-off frequencies below which photons will not propagate in the system are determined by the solutions of

$$\frac{\omega_p^2}{\omega^2} = \frac{\omega \pm \omega_c \pm \hbar q^2/2m}{\omega \pm \hbar q^2/2m}, \quad (27)$$

where '+' applies to right circular polarized modes while '−' applies to left circular polarized modes. For left circularly polarized modes equation (27) yields

$$\omega \approx \omega_c + \hbar q^2/2m + \frac{\omega_p^2\,(\omega_c + \hbar q^2/2m) - \hbar q^2\omega_p^2/2m}{(\omega_c + \hbar q^2/2m)^2 - \omega_p^2} + \cdots\,. \quad (28)$$

The dielectric response functions for photon propagation perpendicular to the magnetic field have been presented in terms of the Bose Kummer function in reference [19], but in terms of $S(\alpha,x)$ they become

$$\epsilon_1(q,\omega) = 1 + \frac{\omega_p^2}{2z_*\,\omega_c^2}\,e^{-z_*} \times\Big[S\left(\omega/\omega_c, z_*\right) + S\left(-\omega/\omega_c, z_*\right)\Big], \quad (29)$$

$$\epsilon_2(q,\omega) = 1 - \frac{2\omega_{\rm p}^2}{\omega^2} + \frac{\omega_{\rm p}^2\,{\rm e}^{-z_*}}{2\omega^2 z_*\omega_{\rm c}^2}\Big[(z_*\omega_{\rm c}+\omega)^2 S\left(\omega/\omega_{\rm c}, z_*\right) + (z_*\omega_{\rm c}-\omega)^2 S\left(-\omega/\omega_{\rm c}, z_*\right)\Big], \quad (30)$$

$$\epsilon_3(q,\omega) = 1 - \omega_{\rm p}^2/\omega^2, \quad (31)$$

and

$$\epsilon_x(q,\omega) = \frac{\omega_{\rm p}^2}{2z_*\omega_{\rm c}^2\,\omega}\,{\rm e}^{-z_*}\Big[(z_*\,\omega_{\rm c}+\omega)\,S\left(\omega/\omega_{\rm c}, z_*\right) - (z_*\,\omega_{\rm c}-\omega)\,S\left(-\omega/\omega_{\rm c}, z_*\right)\Big], \quad (32)$$

where $z_* = \hbar q^2/2m\omega_{\rm c}$. The first three dielectric response functions appear as diagonal components while the last one is the sole off-diagonal component of the dielectric tensor. The dispersion relation for the ordinary mode is obtained by setting equation (8) equal to q^2c^2/ω^2, which yields $\omega^2 = \omega_{\rm p}^2 + q^2c^2$. Thus this mode is unaffected by the magnetic field, which is identical to the situation for a classical magnetoplasma in the cold plasma limit [14]. Unlike the classical magnetoplasma, however, $\epsilon_2(q,\omega)$ does not reduce $\epsilon_1(q,\omega)$, whose origin can be traced back to the diagonal components of equation (38) of reference [16].

If equation (7) is introduced into equations (29, 30, 32), then the various dielectric response functions for photon propagation perpendicular to the magnetic field become

$$\epsilon_1(q,\omega) \sim 1 + \frac{\omega_{\rm p}^2}{\omega_{\rm c}^2}\left[\frac{(1-z_*)}{D(z_*,1,\omega/\omega_{\rm c})} + \frac{2z_*(z_*+1)}{D(z_*,1,\omega/\omega_{\rm c})^2} + \frac{z_*}{D(z_*,2,\omega/\omega_{\rm c})} + \cdots\right], \quad (33)$$

$$\begin{aligned}\epsilon_2(q,\omega) \sim 1 &- \frac{2\omega_{\rm p}^2}{\omega^2}\left(1-{\rm e}^{-z_*}\right) + \frac{\omega_{\rm p}^2}{\omega^2}\left[\frac{1-3z_*+z_*/(z_*+1)}{D(z_*,1,\omega/\omega_{\rm c})} + \frac{2z_*}{D(z_*,2,\omega/\omega_{\rm c})}\right.\\ &+ \frac{z_*(z_*^2+1)}{(z_*+1)D(z_*,1,\omega/\omega_{\rm c})^2} - \frac{1-3z_*+z_*^2}{z_*+1}\\ &\left.+\frac{z_*(z_*-1)}{(z_*+1)^2} + \frac{z_*(z_*-2)}{z_*+2} + \cdots\right], \quad (34)\end{aligned}$$

and

$$\begin{aligned}\epsilon_x(q,\omega) \sim \frac{\omega_{\rm p}^2}{\omega^2}\,{\rm e}^{-z_*} &+ \frac{\omega_{\rm p}^2}{\omega_{\rm c}^2}\left[\frac{1-2z_*+z_*/(z_*+1)}{(z_*+1)D(z_*,1,\omega/\omega_{\rm c})} + \frac{2z_*}{D(z_*,1,\omega/\omega_{\rm c})^2}\right.\\ &\left.+\frac{2z_*}{(z_*+2)D(z_*,2,\omega/\omega_{\rm c})} + \cdots\right], \quad (35)\end{aligned}$$

where $D(x,n,z) = (x+n)^2 - z^2$. Strictly speaking, the exponential factor of ${\rm e}^{-z_*}$ should be expanded in powers of z_*, which is done from here on.

The dispersion relation for the extraordinary mode is determined from

$$q^2c^2/\omega^2 = (\epsilon_1\epsilon_2 - \epsilon_x^2)/\epsilon_1. \quad (36)$$

To first order term in z_* equation (13) becomes

$$\begin{aligned}&\left(1 + \frac{\omega_{\rm p}^2}{\omega_{\rm c}^2}\left(\frac{1-z_*}{D(z_*,1,\omega/\omega_{\rm c})} + \frac{2z_*}{D(z_*,1,\omega/\omega_{\rm c})^2} + \frac{z_*}{D(z_*,2,\omega/\omega_{\rm c})}\right)\right)\\ &\times\left(1 - \frac{2z_*\omega_{\rm p}^2}{\omega^2} - \frac{q^2c^2}{\omega^2} + \frac{\omega_{\rm p}^2}{\omega_{\rm c}^2}\left(\frac{1-3z_*}{D(z_*,1,\omega/\omega_{\rm c})} + \frac{4z_*}{D(z_*,2,\omega/\omega_{\rm c})} + \frac{2z_*}{D(z_*,1,\omega/\omega_{\rm c})^2} - 1 + 2z_*\right)\right)\\ &= \left(\frac{\omega_{\rm p}^2(1-z_*)}{\omega^2} + \frac{\omega_{\rm p}^2}{\omega_{\rm c}^2}\left(\frac{1-2z_*}{D(z_*,1,\omega/\omega_{\rm c})} + \frac{z_*}{D(z_*,2,\omega/\omega_{\rm c})} + \frac{2z_*}{D(z_*,1,\omega/\omega_{\rm c})^2}\right)\right)^2. \quad (37)\end{aligned}$$

This equation is best solved numerically, but since $|z_*|$ is very small, we can evaluate the dominant contributions to the dispersion relations by studying the equation to zeroth order in z_*. Before doing so, however, we need to determine the conditions under which the terms of non-zeroth order in z_* can be neglected. This is necessary because the limit $z_* \to 0$ means that $\omega_{\rm c} \to \infty$ and some of the non-zeroth order terms in z_* in equation (37) contain powers of $\omega_{\rm c}$, *e.g.* the last term. To neglect these terms, one needs to ensure that the denominators are large, which is valid as long as ω is not close to the first harmonic of the cyclotron frequency. However, for $|\omega_{\rm p}/\omega_{\rm c}| \ll 1$ it is possible to obtain a dispersion relation near the fundamental cyclotron frequency as found for circularly polarized waves, *viz.* equation (26). If it is assumed that in the vicinity of the cyclotron frequency that the dispersion relation will be of the form, $\omega^2 = \omega_{\rm c}^2 + \beta_1\omega_{\rm p}^2 + ...$, where $\beta_1 = O(1)$, then the condition that such terms as the last term in equation (37) can be neglected is $z_* \ll \omega_{\rm p}^2/\omega_{\rm c}^2$, placing a constraint on the size of the wavenumber q.

The constraint on the wavenumber does not mean, however, that q must always be small. For a magnetic field of the 10^5 G, though much stronger fields are discussed in relation to high temperature superconductors in reference [10], one obtains a cyclotron frequency of 1.8×10^{12} rad s^{-1}, which means for a plasma frequency of 10^{11} rad s^{-1} that $\omega_{\rm p}^2/\omega_{\rm c}^2 = 1/324$. Then the condition on z_* means that $q^2 \ll 1.4 \times 10^5$ cm^{-1}. Thus, corrections involving z_* in equation (37) will generally be extremely small and can be neglected.

To zeroth order order in z_*, equation (37) reduces to

$$q^2c^2\omega^2\left(\omega_c^2-\omega^2+\omega_p^2\right)= \\ =\left(\omega^2-\omega_p^2\right)\left(\omega^2\omega_c^2-\omega^4+\omega_p^2\omega^2+\omega_p^2\omega_c^2\right)\ . \quad (38)$$

The solutions to this equation can be obtained by carrying out a perturbational analysis around the solutions obtained by setting the lhs equal to zero. Although all solutions for ω^2 are real, one is negatively real, which, in turn, means that ω is imaginary. This branch is neglected. The two remaining solutions for ω^2 are then perturbed around $\omega^2=\omega_c^2$ and $\omega^2=\omega_p^2$. For large $|\omega_c|$, the solution corresponding to close to the cyclotron frequency is given by

$$\omega^2\approx\omega_c^2+\omega_p^2\left(1+\beta_0\right)-\frac{\omega_p^4(1+\beta_0^2)}{(\omega_c^2-q^2c^2)}+\cdots\ , \quad (39)$$

where $\beta_0=\omega_c^2/(\omega_c^2-q^2c^2)$ and $\omega_p^2\ll\omega_c^2$. Close to the plasma frequency equation (16) yields

$$\omega^2\approx\omega_p^2+\frac{q^2c^2}{2+q^2c^2\left(\omega_c^{-2}-\omega_p^{-2}\right)}+\cdots\ , \quad (40)$$

which has been obtained under the condition that $q^2c^2\ll\omega_p^2$.

Resonances for the extraordinary mode occur whenever qc/ω is infinite, which in turn means that $\epsilon_1(q,\omega)=0$. There is no need to solve this equation directly because under resonance conditions $\epsilon_1(q,\omega)$ equals the longitudinal dielectric response function with q_z set equal to zero. Thus resonance conditions correspond to solving the longitudinal dielectric response function set equal to zero, which we have seen already yields the dispersion relation for Bernstein mode near the cyclotron frequency, *viz.* equation (10).

Cut-off frequencies are determined by setting the lhs of equation (38) equal to zero. When this is done, two frequencies are found: the first cut-off frequency occurring not unexpectedly at $\omega^2=\omega_p^2$ while the second or upper cut-off frequency occurs at

$$\omega^2=\frac{1}{2}\left(\omega_c^2+\omega_p^2\right)+\frac{1}{2}\sqrt{\left(\omega_c^2+\omega_p^2\right)^2+4\omega_p^2\omega_c^2}. \quad (41)$$

An extraordinary mode will only propagate if its frequency is greater than the plasma frequency. On the other hand, for $\omega_p^2\ll\omega_c^2$, the upper cut-off frequency corresponds to $\omega^2\approx\omega_c^2+2\omega_p^2$. Extraordinary modes will only propagate between the two cut-off frequencies with the greatest absorption by the medium of these modes occurring at the Bernstein mode given by equation (12) in reference [4].

4 Conclusion

In this paper we have presented a summary of the main dielectric properties of an ultra-cold strongly magnetized CBG. In particular, we have given the dispersion relations for longitudinal modes propagating parallel to the magnetic field, the so-called plasmon modes given by equation (1), and for those propagating perpendicular to the magnetic field, which are known as Bernstein modes. Bernstein modes arise near the the cyclotron frequency as given by equation (10) or close to harmonics of this frequency such as equations (11, 12). In addition, the general anisotropic case for longitudinal modes has been studied, which represents a hybrid mode as given by equation (16).

The dispersion relations for tranverse modes propagating both parallel and perpendicular to the magnetic field have also been presented. The dispersion relations for circularly polarized modes are given by equation (25) with the solution near the cyclotron frequency given by equation (26). Left circularly polarized modes were found to exhibit resonance at $\omega=\omega_c+\hbar q^2/2m$ while their cut-off frequency is given by equation (28). The dispersion relations for extraordinary modes near the cyclotron frequency and near the plasma frequency are given by equations (39, 40) respectively. The resonance condition for this hybrid mode was found to be given by the dispersion relation for the Bernstein mode near the cyclotron frequency, *i.e.*, equation (10), while the cut-off frequencies were found to occur at the plasma frequency and at the higher frequency given by equation (41).

A more detailed study incorporating numerical analyses of these results is currently being carried out. It should also be mentioned that the novel asymptotic expansion for $S(\alpha,x)$ given here can also be applied to evaluate the strong magnetic field behaviour for an even more important fundamental system of condensed matter, the magnetized degenerate electron gas. This study, too, is currently underway.

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